

# On the Capacity of the $K$ -User Cyclic Gaussian Interference Channel

Lei Zhou, *Student Member, IEEE* and Wei Yu, *Senior Member, IEEE*

**Abstract**—This paper studies the capacity region of a  $K$ -user cyclic Gaussian interference channel, where the  $k$ th user interferes with only the  $(k-1)$ th user (mod  $K$ ) in the network. Inspired by the work of Etkin, Tse and Wang, who derived a capacity region outer bound for the two-user Gaussian interference channel and proved that a simple Han-Kobayashi power splitting scheme can achieve to within one bit of the capacity region for all values of channel parameters, this paper shows that a similar strategy also achieves the capacity region of the  $K$ -user cyclic interference channel to within a constant gap in the weak interference regime. Specifically, for the  $K$ -user cyclic Gaussian interference channel, a compact representation of the Han-Kobayashi achievable rate region using Fourier-Motzkin elimination is first derived, a capacity region outer bound is then established. It is shown that the Etkin-Tse-Wang power splitting strategy gives a constant gap of at most 2 bits in the weak interference regime. For the special 3-user case, this gap can be sharpened to  $1\frac{1}{2}$  bits by time-sharing of several different strategies. The capacity result of the  $K$ -user cyclic Gaussian interference channel in the strong interference regime is also given. Further, based on the capacity results, this paper studies the generalized degrees of freedom (GDoF) of the symmetric cyclic interference channel. It is shown that the GDoF of the symmetric capacity is the same as that of the classic two-user interference channel, no matter how many users are in the network.

**Index Terms**—Approximate capacity, Han-Kobayashi, Fourier-Motzkin,  $K$ -user interference channel, multicell processing.

## I. INTRODUCTION

The interference channel models a communication scenario in which several mutually interfering transmitter-receiver pairs share the same physical medium. The interference channel is a useful model for many practical systems such as the wireless network. The capacity region of the interference channel, however, has not been completely characterized, even for the two-user Gaussian case.

The largest known achievable rate region for the two-user interference channel is given by Han and Kobayashi [1] using a coding scheme involving common-private power splitting. Chong et al. [2] obtained the same rate region in a simpler form by applying the Fourier-Motzkin algorithm together with

a time-sharing technique to the Han and Kobayashi's rate region characterization. The optimality of the Han-Kobayashi region for the two-user Gaussian interference channel is still an open problem in general, except in the strong interference regime where transmission with common information only achieves the capacity region [1], [3], [4], and in a noisy interference regime where transmission with private information only achieves the sum capacity [5]–[7].

In a breakthrough, Etkin, Tse and Wang [8] showed that the Han-Kobayashi scheme can in fact achieve to within one bit of the capacity region for the two-user Gaussian interference channel for all channel parameters. Their key insight was that the interference-to-noise ratio (INR) of the private message should be chosen to be as close to 1 as possible in the Han-Kobayashi scheme. They also found a new capacity region outer bound using a genie-aided technique. In the rest of this paper, we refer this particular setting of the private message power as the Etkin-Tse-Wang (ETW) power-splitting strategy.

The Etkin, Tse and Wang's result applies only to the two-user interference channel. Practical systems often have more than two transmitter-receiver pairs, yet the generalization of Etkin, Tse and Wang's work to the interference channels with more than two users has proved difficult for the following reasons. First, it appears that the Han-Kobayashi common-private superposition coding is no longer adequate for the  $K$ -user interference channel. Interference alignment types of coding scheme [9] [10] can potentially enlarge the achievable rate region. Second, even within the Han-Kobayashi framework, when more than two receivers are involved, multiple common messages at each transmitter may be needed, making the optimization of the resulting rate region difficult.

In the context of  $K$ -user Gaussian interference channels, sum capacity results are available in the noisy interference regime [5], [11]. In particular, Annapureddy et al. [5] obtained the sum capacity for the symmetric three-user Gaussian interference channel, the one-to-many, and the many-to-one Gaussian interference channels under the noisy interference criterion. Similarly, Shang et al. [11] studied the fully connected  $K$ -user Gaussian interference channel and showed that treating interference as noise at the receiver is sum-capacity achieving when the transmit power and the cross channel gains are sufficiently weak to satisfy a certain criterion. Further, achievability and outer bounds for the three-user interference channel have also been studied in [12] and [13]. Finally, much work has been carried out on the generalized degree of freedom (GDoF as defined in [8]) of the  $K$ -user interference channel and its variations [9], [14]–[16].

Instead of treating the general  $K$ -user interference channel, this paper focuses on a cyclic Gaussian interference channel,

Manuscript received October 4, 2010; revised May 8, 2012; accepted August 28, 2012. Date of current version August 30, 2012. This work was supported by the Natural Science and Engineering Research Council (NSERC). The material in this paper was presented in part at the 2010 IEEE Conference on Information Science and Systems (CISS), and in part at the 2011 IEEE Symposium of Information Theory (ISIT).

The authors are with The Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON M5S 3G4 Canada (email: zhoulei@comm.utoronto.ca; weiyu@comm.utoronto.ca). Kindly address correspondence to Lei Zhou (zhoulei@comm.utoronto.ca).

Copyright (c) 2012 IEEE. Personal use of this material is permitted. However, permission to use this material for any other purposes must be obtained from the IEEE by sending a request to pubs-permissions@ieee.org.

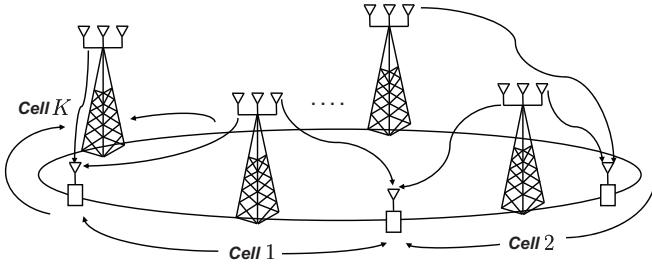


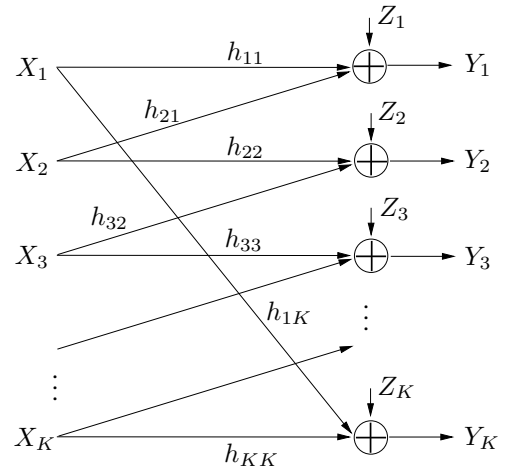
Fig. 1. The circular array soft-handoff model

where the  $k$ th user interferes with only the  $(k - 1)$ th user. In this case, each transmitter interferes with only one other receiver, and each receiver suffers interference from only one other transmitter, thereby avoiding the difficulties mentioned earlier. For the  $K$ -user cyclic interference channel, the Etkin, Tse and Wang's coding strategy remains a natural one. The main objective of this paper is to show that it indeed achieves to within a constant gap of the capacity region for this cyclic channel in the weak interference regime to be defined later.

The cyclic interference channel model is motivated by the so-called modified Wyner model, which describes the soft handoff scenario of a cellular network [17]. The original Wyner model [18] assumes that all cells are arranged in a linear array with the base-stations located at the center of each cell, and where intercell interference comes from only the two adjacent cells. In the modified Wyner model [17] cells are arranged in a circular array as shown in Fig. 1. The mobile terminals are located along the circular array. If one assumes that the mobile terminals always communicate with the intended base-station to their left (or right), while only suffering from interference due to the base-station to their right (or left), one arrives at the  $K$ -user cyclic Gaussian interference channel studied in this paper. The modified Wyner model has been extensively studied in the literature [17], [19], [20], but often either with interference treated as noise or with the assumption of full base-station cooperation. This paper studies the modified Wyner model without base-station cooperation, in which case the soft-handoff problem becomes that of a cyclic interference channel.

This paper primarily focuses on the  $K$ -user cyclic Gaussian interference channel in the weak interference regime. The main contributions of this paper are as follows. This paper first derives a compact characterization of the Han-Kobayashi achievable rate region by applying the Fourier-Motzkin elimination algorithm. A capacity region outer bound is then obtained. It is shown that with the Etkin, Tse and Wang's coding strategy, one can achieve to within  $1\frac{1}{2}$  bits of the capacity region when  $K = 3$  (with time-sharing), and to within two bits of the capacity region in general in the weak interference regime. Finally, the capacity result for the strong interference regime is also derived.

A key part of the development involves a Fourier-Motzkin elimination procedure on the achievable rate region of the  $K$ -user cyclic interference channel. To deal with the large number of inequality constraints, an induction proof is used. It is shown that as compared to the two-user case, where the

Fig. 2.  $K$ -user cyclic Gaussian interference channel

rate region is defined by constraints on the individual rate  $R_i$ , the sum rate  $R_1 + R_2$ , and the sum rate plus an individual rate  $2R_i + R_j$  ( $i \neq j$ ), the achievable rate region for the  $K$ -user cyclic interference channel is defined by an additional set of constraints on the sum rate of any arbitrary  $l$  adjacent users, where  $2 \leq l < K$ . These four types of rate constraints completely characterize the Han-Kobayashi region for the  $K$ -user cyclic interference channel. They give rise to a total of  $K^2 + 1$  constraints.

For the symmetric  $K$ -user cyclic channel where all direct links share the same channel gain and all cross links share another channel gain, it is shown that the GDoF of the symmetric capacity is not dependent on the number of users in the network. Therefore, adding more users to a  $K$ -user cyclic interference channel with symmetric channel parameters does not affect the per-user rate.

## II. CHANNEL MODEL

The  $K$ -user cyclic Gaussian interference channel (as depicted in Fig. 2) has  $K$  transmitter-receiver pairs. Each transmitter tries to communicate with its intended receiver while causing interference to only one neighboring receiver. Each receiver receives a signal intended for it and an interference signal from only one neighboring sender plus an additive white Gaussian noise (AWGN). As shown in Fig. 2,  $X_1, X_2, \dots, X_K$  and  $Y_1, Y_2, \dots, Y_K$  are the complex-valued input and output signals, respectively, and  $Z_i \sim \mathcal{CN}(0, \sigma^2)$  is the independent and identically distributed (i.i.d) Gaussian noise at receiver  $i$ . The input-output model can be written as

$$\begin{aligned} Y_1 &= h_{1,1}X_1 + h_{2,1}X_2 + Z_1, \\ Y_2 &= h_{2,2}X_2 + h_{3,2}X_3 + Z_2, \\ &\vdots \\ Y_K &= h_{K,K}X_K + h_{1,K}X_1 + Z_K, \end{aligned}$$

where each  $X_i$  has a power constraint  $P_i$  associated with it, i.e.,  $\mathbb{E}[|x_i|^2] \leq P_i$ . Here,  $h_{i,j}$  is the channel gain from transmitter  $i$  to receiver  $j$ .

Define the signal-to-noise and interference-to-noise ratios for each user as follows:

$$\text{SNR}_i = \frac{|h_{i,i}|^2 P_i}{\sigma^2} \quad \text{INR}_i = \frac{|h_{i,i-1}|^2 P_i}{\sigma^2}, \quad i = 1, 2, \dots, K. \quad (1)$$

The  $K$ -user cyclic Gaussian interference channel is said to be in the *weak* interference regime if

$$\text{INR}_i \leq \text{SNR}_i, \quad \forall i = 1, 2, \dots, K. \quad (2)$$

and the *strong* interference regime if

$$\text{INR}_i \geq \text{SNR}_i, \quad \forall i = 1, 2, \dots, K. \quad (3)$$

Otherwise, it is said to be in the *mixed* interference regime. Throughout this paper, modulo arithmetic is implicitly used on the user indices, e.g.,  $K+1 = 1$  and  $1-1 = K$ . Note that when  $K = 2$ , the cyclic channel reduces to the conventional two-user interference channel.

### III. WITHIN TWO BITS OF THE CAPACITY REGION IN THE WEAK INTERFERENCE REGIME

The generalization of Etkin, Tse and Wang's result to the capacity region of a general (nonsymmetric)  $K$ -user cyclic Gaussian interference channel is significantly more complicated. In the two-user case, the shape of the Han-Kobayashi achievable rate region is the union of polyhedrons (each corresponding to a fixed input distribution) with boundaries defined by rate constraints on  $R_1$ ,  $R_2$ ,  $R_1 + R_2$ ,  $2R_1 + R_2$  and  $2R_2 + R_1$ , respectively. In the multiuser case, to extend Etkin, Tse and Wang's result, one needs to find a similar rate region characterization for the general  $K$ -user cyclic interference channel first.

A key feature of the cyclic Gaussian interference channel model is that each transmitter sends signal to its intended receiver while causing interference to *only one* of its neighboring receivers; meanwhile, each receiver receives the intended signal plus the interfering signal from *only one* of its neighboring transmitters. Using this fact and with the help of Fourier-Motzkin elimination algorithm, this section shows that the achievable rate region of the  $K$ -user cyclic Gaussian interference channel is the union of polyhedrons with boundaries defined by rate constraints on the individual rates  $R_i$ , the sum rate  $R_{\text{sum}}$ , the sum rate plus an individual rate  $R_{\text{sum}} + R_i$  ( $i = 1, 2, \dots, K$ ), and the sum rate for arbitrary  $l$  adjacent users ( $2 \leq l < K$ ). This last rate constraint on arbitrary  $l$  adjacent users' rates is new as compared with the two-user case.

The preceding characterization together with outer bounds to be proved later in the section allows us to prove that the capacity region of the  $K$ -user cyclic Gaussian interference channel can be achieved to within a constant gap using the ETW power-splitting strategy in the weak interference regime. However, instead of the one-bit result for the two-user interference channel, this section shows that one can achieve to within  $1\frac{1}{2}$  bits of the capacity region when  $K = 3$  (with time-sharing), and within two bits of the capacity region for general  $K$ . Again, the strong interference regime is treated later.

#### A. Achievable Rate Region

**Theorem 1:** Let  $\mathcal{P}$  denote the set of probability distributions  $P(\cdot)$  that factor as

$$P(q, w_1, x_1, w_2, x_2, \dots, w_K, x_K) = p(q)p(x_1 w_1 | q)p(x_2 w_2 | q) \cdots p(x_K w_K | q). \quad (4)$$

For a fixed  $P \in \mathcal{P}$ , let  $\mathcal{R}_{\text{HK}}^{(K)}(P)$  be the set of all rate tuples  $(R_1, R_2, \dots, R_K)$  satisfying

$$0 \leq R_i \leq \min\{d_i, a_i + e_{i-1}\}, \quad (5)$$

$$\sum_{j=m}^{m+l-1} R_j \leq \min \left\{ g_m + \sum_{j=m+1}^{m+l-2} e_j + a_{m+l-1}, \sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1} \right\}, \quad (6)$$

$$\sum_{j=1}^K R_j \leq \min \left\{ \sum_{j=1}^K e_j, r_1, r_2, \dots, r_K \right\}, \quad (7)$$

$$\sum_{j=1}^K R_j + R_i \leq a_i + g_i + \sum_{j=1, j \neq i}^K e_j, \quad (8)$$

where  $a_i, d_i, e_i, g_i$  and  $r_i$  are defined as follows:

$$a_i = I(Y_i; X_i | W_i, W_{i+1}, Q) \quad (9)$$

$$d_i = I(Y_i; X_i | W_{i+1}, Q) \quad (10)$$

$$e_i = I(Y_i; W_{i+1}, X_i | W_i, Q) \quad (11)$$

$$g_i = I(Y_i; W_{i+1}, X_i | Q) \quad (12)$$

$$r_i = a_{i-1} + g_i + \sum_{j=1, j \notin \{i, i-1\}}^K e_j, \quad (13)$$

and the range of indices are  $i, m = 1, 2, \dots, K$  in (5) and (8),  $l = 2, 3, \dots, K-1$  in (6). Define

$$\mathcal{R}_{\text{HK}}^{(K)} = \bigcup_{P \in \mathcal{P}} \mathcal{R}_{\text{HK}}^{(K)}(P). \quad (14)$$

Then  $\mathcal{R}_{\text{HK}}^{(K)}$  is an achievable rate region for the  $K$ -user cyclic interference channel<sup>1</sup>.

**Proof:** The achievable rate region can be proved by the Fourier-Motzkin algorithm together with an induction step. The proof follows the Kobayashi and Han's strategy [22] of eliminating a common message at each step. The details are presented in Appendix A. ■

In the above achievable rate region, (5) is the constraint on the achievable rate of an individual user, (6) is the constraint on the achievable sum rate for any  $l$  adjacent users ( $2 \leq l < K$ ), (7) is the constraint on the achievable sum rate of all  $K$  users, and (8) is the constraint on the achievable sum rate for all  $K$  users plus a repeated one. We can also think of (5)-(8) as the sum-rate constraints for arbitrary  $l$  adjacent users, where  $l = 1$  for (5),  $2 \leq l < K$  for (6),  $l = K$  for (7) and  $l = K+1$  for (8).

From (5) to (8), there are a total of  $K + K(K-2) + 1 + K = K^2 + 1$  constraints. Together they describe the shape of the

<sup>1</sup>The same achievable rate region has been found independently in [21].

achievable rate region under a fixed input distribution. The quadratic growth in the number of constraints as a function of  $K$  makes the Fourier-Motzkin elimination of the Han-Kobayashi region quite complex. The proof in Appendix A uses induction to deal with the large number of the constraints.

As an example, for the two-user Gaussian interference channel, there are  $2^2 + 1 = 5$  rate constraints, corresponding to that of  $R_1$ ,  $R_2$ ,  $R_1 + R_2$ ,  $2R_1 + R_2$  and  $2R_2 + R_1$ , as in [1], [2], [8], [22]. Specifically, substituting  $K = 2$  in Theorem 1 gives us the following achievable rate region:

$$0 \leq R_1 \leq \min\{d_1, a_1 + e_2\}, \quad (15)$$

$$0 \leq R_2 \leq \min\{d_2, a_2 + e_1\}, \quad (16)$$

$$R_1 + R_2 \leq \min\{e_1 + e_2, a_1 + g_2, a_2 + g_1\}, \quad (17)$$

$$2R_1 + R_2 \leq a_1 + g_1 + e_2, \quad (18)$$

$$2R_2 + R_1 \leq a_2 + g_2 + e_1. \quad (19)$$

The above region for the two-user Gaussian interference channel is exactly that of Theorem D in [22].

### B. Capacity Region Outer Bound

*Theorem 2:* For the  $K$ -user cyclic Gaussian interference channel in the weak interference regime, the capacity region is included in the set of rate tuples  $(R_1, R_2, \dots, R_K)$  such that

$$R_i \leq \lambda_i, \quad (20)$$

$$\sum_{j=m}^{m+l-1} R_j \leq \min \left\{ \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1}, \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right\}, \quad (21)$$

$$\sum_{j=1}^K R_j \leq \min \left\{ \sum_{j=1}^K \alpha_j, \rho_1, \rho_2, \dots, \rho_K \right\}, \quad (22)$$

$$\sum_{j=1}^K R_j + R_i \leq \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j, \quad (23)$$

where the ranges of the indices  $i$ ,  $m$ ,  $l$  are as defined in Theorem 1, and

$$\alpha_i = \log \left( 1 + \text{INR}_{i+1} + \frac{\text{SNR}_i}{1 + \text{INR}_i} \right) \quad (24)$$

$$\beta_i = \log \left( \frac{1 + \text{SNR}_i}{1 + \text{INR}_i} \right) \quad (25)$$

$$\gamma_i = \log(1 + \text{INR}_{i+1} + \text{SNR}_i) \quad (26)$$

$$\lambda_i = \log(1 + \text{SNR}_i) \quad (27)$$

$$\mu_i = \log(1 + \text{INR}_i) \quad (28)$$

$$\rho_i = \beta_{i-1} + \gamma_i + \sum_{j=1, j \notin \{i, i-1\}}^K \alpha_j. \quad (29)$$

*Proof:* See Appendix B. ■

### C. Capacity Region to Within Two Bits

*Theorem 3:* For the  $K$ -user cyclic Gaussian interference channel in the weak interference regime, the fixed ETW power-splitting strategy achieves to within two bits of the capacity region<sup>2</sup>.

*Proof:* Applying the ETW power-splitting strategy (i.e.,  $\text{INR}_{ip} = \min(\text{INR}_i, 1)$ ) to Theorem 1, parameters  $a_i, d_i, e_i, g_i$  can be easily calculated as follows:

$$a_i = \log(2 + \text{SNR}_{ip}) - 1, \quad (30)$$

$$d_i = \log(2 + \text{SNR}_i) - 1, \quad (31)$$

$$e_i = \log(1 + \text{INR}_{i+1} + \text{SNR}_{ip}) - 1, \quad (32)$$

$$g_i = \log(1 + \text{INR}_{i+1} + \text{SNR}_i) - 1, \quad (33)$$

where  $\text{SNR}_{ip} = |h_{i,i}|^2 P_{ip} / \sigma^2$ . To prove that the achievable rate region in Theorem 1 with the above  $a_i, d_i, e_i, g_i$  is within two bits of the outer bound in Theorem 2, we show that each of the rate constraints in (5)-(8) is within two bits of their corresponding outer bound in (20)-(23) in the weak interference regime, i.e., the following inequalities hold for all  $i, m, l$  in the ranges defined in Theorem 1:

$$\delta_{R_i} \leq 2, \quad (34)$$

$$\delta_{R_m + \dots + R_{m+l-1}} \leq 2l, \quad (35)$$

$$\delta_{R_{\text{sum}}} \leq 2K, \quad (36)$$

$$\delta_{R_{\text{sum}} + R_i} \leq 2(K+1), \quad (37)$$

where  $\delta_{(\cdot)}$  is the difference between the achievable rate in Theorem 1 and its corresponding outer bound in Theorem 2. The proof makes use of a set of inequalities provided in Appendix D.

For  $\delta_{R_i}$ , we have

$$\begin{aligned} \delta_{R_i} &= \lambda_i - \min\{d_i, a_i + e_{i-1}\} \\ &= \max\{\lambda_i - d_i, \lambda_i - (a_i + e_{i-1})\} \\ &\leq 2. \end{aligned} \quad (38)$$

For  $\delta_{R_m + \dots + R_{m+l-1}}$ , compare the first terms of (6) and (21):

$$\begin{aligned} \delta_1 &= \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1} - g_m + \sum_{j=m+1}^{m+l-2} e_j \\ &\quad + a_{m+l-1} \\ &= (\gamma_m - g_m) + \sum_{j=m+1}^{m+l-2} (\alpha_j - e_j) + (\beta_{m+l-1} - a_{m+l-1}) \\ &\leq l. \end{aligned} \quad (39)$$

Similarly, the difference between the second term of (6) and

<sup>2</sup>This paper follows the definition from [8] that if a rate tuple  $(R_1, R_2, \dots, R_K)$  is achievable and  $(R_1 + b, R_2 + b, \dots, R_K + b)$  is outside the capacity region, then  $(R_1, R_2, \dots, R_K)$  is within  $b$  bits of the capacity region.

(21) is bounded by

$$\begin{aligned}
\delta_2 &= \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} - \sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1} \\
&= (\mu_m - e_{m-1}) + \sum_{j=m}^{m+l-2} (\alpha_j - e_j) \\
&\quad + (\beta_{m+l-1} - a_{m+l-1}) \\
&\leq l + 1.
\end{aligned} \tag{40}$$

Finally, applying the fact that

$$\min\{x_1, y_1\} - \min\{x_2, y_2\} \leq \max\{x_1 - x_2, y_1 - y_2\},$$

we obtain

$$\delta_{R_m + \dots + R_{m+l-1}} \leq \max\{\delta_1, \delta_2\} \leq l + 1. \tag{41}$$

For  $\delta_{R_{sum}}$ , the difference between the first terms of (7) and (22) is bounded by

$$\sum_{j=1}^K \alpha_j - \sum_{j=1}^K e_j = \sum_{j=1}^K (\alpha_j - e_j) \leq K. \tag{42}$$

In addition,

$$\begin{aligned}
\rho_i - r_i &= \beta_{i-1} + \gamma_i + \sum_{j=1, j \notin \{i, i-1\}}^K \alpha_j \\
&\quad - a_{i-1} + g_i + \sum_{j=1, j \notin \{i, i-1\}}^K e_j \\
&= (\beta_{i-1} - a_{i-1}) + (\gamma_i - g_i) \\
&\quad + \sum_{j=1, j \notin \{i, i-1\}}^K (\alpha_j - e_j) \\
&\leq K
\end{aligned} \tag{43}$$

for  $i = 1, 2, \dots, K$ . As a result, the gap on the sum rate is bounded by

$$\begin{aligned}
\delta_{R_{sum}} &= \min \left\{ \sum_{j=1}^K \alpha_j, \rho_1, \rho_2, \dots, \rho_K \right\} \\
&\quad - \min \left\{ \sum_{j=1}^K e_j, r_1, r_2, \dots, r_K \right\} \\
&\leq \max \left\{ \sum_{j=1}^K (\alpha_j - e_j), \rho_1 - r_1, \right. \\
&\quad \left. \rho_2 - r_2, \dots, \rho_K - r_K \right\} \\
&\leq K.
\end{aligned} \tag{44}$$

For  $R_{sum} + R_i$ , we have

$$\begin{aligned}
\delta_{R_{sum} + R_i} &= \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j - a_i + g_i + \sum_{j=1, j \neq i}^K e_j \\
&= (\beta_i - a_i) + (\gamma_i - g_i) + \sum_{j=1, j \neq i}^K (\alpha_j - e_j) \\
&\leq K + 1
\end{aligned} \tag{45}$$

Since the inequalities in (34)-(37) hold for all the ranges of  $i$ ,  $m$ , and  $l$  defined in Theorem 1, this proves that the ETW power-splitting strategy achieves to within two bits of the capacity region in the weak interference regime. ■

#### D. 3-User Cyclic Gaussian Interference Channel Capacity Region to Within $1\frac{1}{2}$ Bits

Chong, Motani and Garg [2] showed that by time-sharing with marginalized versions of the input distribution, the Han-Kobayashi region for the two-user interference channel as stated in (15)-(19) can be further simplified by removing the  $a_1 + e_2$  and  $a_2 + e_1$  terms from (15) and (16) respectively. The resulting rate region without these two terms is proved to be equivalent to the original Han-Kobayashi region (15)-(19).

This section shows that the aforementioned time-sharing technique can be applied to the 3-user cyclic interference channel (but not to  $K \geq 4$ ). By a similar time-sharing strategy, the second rate constraint on  $R_1, R_2$  and  $R_3$  can be removed, and the achievable rate region can be shown to be within  $1\frac{1}{2}$  bits of the capacity region in the weak interference regime.

*Theorem 4:* Let  $\mathcal{P}_3$  denote the set of probability distributions  $P_3(\cdot)$  that factor as

$$\begin{aligned}
P_3(q, w_1, x_1, w_2, x_2, w_3, x_3) \\
= p(q)p(x_1 w_1 | q)p(x_2 w_2 | q)p(x_3 w_3 | q).
\end{aligned} \tag{46}$$

For a fixed  $P_3 \in \mathcal{P}_3$ , let  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$  be the set of all rate tuples  $(R_1, R_2, R_3)$  satisfying

$$R_i \leq d_i, \quad i = 1, 2, 3, \tag{47}$$

$$R_1 + R_2 \leq \min\{g_1 + a_2, e_3 + e_1 + a_2\}, \tag{48}$$

$$R_2 + R_3 \leq \min\{g_2 + a_3, e_1 + e_2 + a_3\}, \tag{49}$$

$$R_3 + R_1 \leq \min\{g_3 + a_1, e_2 + e_3 + a_1\}, \tag{50}$$

$$\begin{aligned}
R_1 + R_2 + R_3 &\leq \min\{e_1 + e_2 + e_3, a_3 + g_1 + e_2, \\
&\quad a_1 + g_2 + e_3, a_2 + g_3 + e_1\},
\end{aligned} \tag{51}$$

$$2R_1 + R_2 + R_3 \leq a_1 + g_1 + e_2 + e_3, \tag{52}$$

$$R_1 + 2R_2 + R_3 \leq a_2 + g_2 + e_3 + e_1, \tag{53}$$

$$R_1 + R_2 + 2R_3 \leq a_3 + g_3 + e_1 + e_2, \tag{54}$$

where  $a_i, d_i, e_i, g_i$  are as defined before. Define

$$\mathcal{R}_{\text{HK-TS}}^{(3)} = \bigcup_{P_3 \in \mathcal{P}_3} \mathcal{R}_{\text{HK-TS}}^{(3)}(P_3). \tag{55}$$

Then,  $\mathcal{R}_{\text{HK-TS}}^{(3)}$  is an achievable rate region for the 3-user cyclic Gaussian interference channel. Further, when  $P_3$  is set according to the ETW power-splitting strategy, the rate region  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$  is within  $1\frac{1}{2}$  bits of the capacity region in the weak interference regime.

*Proof:* We prove the achievability of  $\mathcal{R}_{\text{HK-TS}}^{(3)}$  by showing that  $\mathcal{R}_{\text{HK-TS}}^{(3)}$  is equivalent to  $\mathcal{R}_{\text{HK}}^{(3)}$ . First, since  $\mathcal{R}_{\text{HK}}^{(3)}$  contains an extra constraint on each of  $R_1, R_2$  and  $R_3$  (see (5)), it immediately follows that

$$\mathcal{R}_{\text{HK}}^{(3)} \subseteq \mathcal{R}_{\text{HK-TS}}^{(3)}. \tag{56}$$

In Appendix C, it is shown that the inclusion also holds the other way around. Therefore,  $\mathcal{R}_{\text{HK}}^{(3)} = \mathcal{R}_{\text{HK-TS}}^{(3)}$  and as a result,  $\mathcal{R}_{\text{HK-TS}}^{(3)}$  is achievable.

Applying the ETW power-splitting strategy (i.e.,  $\text{INR}_{ip} = \min\{\text{INR}_i, 1\}$  and  $Q$  is fixed) to  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$ , and following along the same line of the proof of Theorem 3, we obtain

$$\delta_{R_i} \leq 1, \quad (57)$$

$$\delta_{R_i+R_{i+1}} \leq 3, \quad (58)$$

$$\delta_{R_{\text{sum}}} \leq 3, \quad (59)$$

$$\delta_{R_{\text{sum}}+R_i} \leq 4, \quad (60)$$

where  $i = 1, 2, 3$ . It then follows that the gap to the capacity region is at most  $1\frac{1}{2}$  bits in the weak interference regime. ■

As shown in Appendix C, the rate region (47)-(54) is obtained by taking the union over the achievable rate regions with input distributions  $P_3, P_3^*, P_3^{**}$  and  $P_3^{***}$ , where  $P_3^*, P_3^{**}$  and  $P_3^{***}$  are the marginalized versions of  $P_3$ . Thus, to achieve within  $1\frac{1}{2}$  bits of the capacity region, one needs to time-share among the ETW power-splitting and its three marginalized variations, rather than using the fixed ETW's input alone.

The key improvement of  $\mathcal{R}_{\text{HK-TS}}^{(3)}$  over  $\mathcal{R}_{\text{HK}}^{(3)}$  is the removal of term  $a_i + e_{i-1}$  in (5) using a time-sharing technique. However, the results in Appendix C hold only for  $K = 3$ . When  $K \geq 4$ , it is easy to verify that  $\mathcal{R}_{\text{HK-TS}}^{(4)}(P_4)$  is not within the union of  $\mathcal{R}_{\text{HK}}^{(4)}(P_4)$  and its marginalized variations, i.e.,  $\mathcal{R}_{\text{HK}}^{(4)} \not\subseteq \mathcal{R}_{\text{HK-TS}}^{(4)}$ . Therefore, the techniques used in this paper only allow the two-bit result to be sharpened to a  $1\frac{1}{2}$ -bit result for the three-user cyclic Gaussian interference channel, but not for  $K \geq 4$ .

#### IV. CAPACITY REGION IN THE STRONG INTERFERENCE REGIME

The results so far pertain only to the weak interference regime, where  $\text{SNR}_i \geq \text{INR}_i, \forall i$ . In the strong interference regime, where  $\text{SNR}_i \leq \text{INR}_i, \forall i$ , the capacity result in [1] [4] for the two-user Gaussian interference channel can be easily extended to the  $K$ -user cyclic case.

*Theorem 5:* For the  $K$ -user cyclic Gaussian interference channel in the strong interference regime, the capacity region is given by the set of  $(R_1, R_2, \dots, R_K)$  such that <sup>3</sup>

$$\begin{cases} R_i \leq \log(1 + \text{SNR}_i) \\ R_i + R_{i+1} \leq \log(1 + \text{SNR}_i + \text{INR}_{i+1}), \end{cases} \quad (61)$$

for  $i = 1, 2, \dots, K$ . In the very strong interference regime where  $\text{INR}_i \geq (1 + \text{SNR}_{i-1})\text{SNR}_i, \forall i$ , the capacity region is the set of  $(R_1, R_2, \dots, R_K)$  with

$$R_i \leq \log(1 + \text{SNR}_i), \quad i = 1, 2, \dots, K. \quad (62)$$

*Proof: Achievability:* It is easy to see that (61) is in fact the intersection of the capacity regions of  $K$  multiple-access channels:

$$\bigcap_{i=1}^K \left\{ (R_i, R_{i+1}) \left| \begin{array}{l} R_i \leq \log(1 + \text{SNR}_i) \\ R_{i+1} \leq \log(1 + \text{INR}_{i+1}) \\ R_i + R_{i+1} \leq \log(1 + \text{SNR}_i + \text{INR}_{i+1}). \end{array} \right. \right\}. \quad (63)$$

Each of these regions corresponds to that of a multiple-access channel with  $W_i^n$  and  $W_{i+1}^n$  as inputs and  $Y_i^n$  as output (with  $U_i^n = U_{i+1}^n = \emptyset$ ). Therefore, the rate region (61) can

be achieved by setting all the input signals to be common messages. This completes the achievability part.

*Converse:* The converse proof follows the idea of [4]. The key ingredient is to show that for a genie-aided Gaussian interference channel to be defined later, in the strong interference regime, whenever a rate tuple  $(R_1, R_2, \dots, R_K)$  is achievable, i.e.,  $X_i^n$  is decodable at receiver  $i$ ,  $X_i^n$  must also be decodable at  $Y_{i-1}^n, i = 1, 2, \dots, K$ .

The genie-aided Gaussian interference channel is defined by the Gaussian interference channel (see Fig. 2) with genie  $X_{i+2}^n$  given to receiver  $i$ . The capacity region of the  $K$ -user cyclic Gaussian interference channel must reside inside the capacity region of the genie-aided one.

Assume that a rate tuple  $(R_1, R_2, \dots, R_K)$  is achievable for the  $K$ -user cyclic Gaussian interference channel. In this case, after  $X_i^n$  is decoded, with the knowledge of the genie  $X_{i+2}^n$ , receiver  $i$  can construct the following signal:

$$\begin{aligned} \tilde{Y}_i^n &= \frac{h_{i+1,i+1}}{h_{i+1,i}}(Y_i^n - h_{i,i}X_i^n) + h_{i+2,i+1}X_{i+2}^n \\ &= h_{i+1,i+1}X_{i+1}^n + h_{i+2,i+1}X_{i+2}^n + \frac{h_{i+1,i+1}}{h_{i+1,i}}Z_i^n, \end{aligned}$$

which contains the signal component of  $Y_{i+1}^n$  but with less noise since  $|h_{i+1,i}| \geq |h_{i+1,i+1}|$  in the strong interference regime. Now, since  $X_{i+1}^n$  is decodable at receiver  $i+1$ , it must also be decodable at receiver  $i$  using the constructed  $\tilde{Y}_i^n$ . Therefore,  $X_i^n$  and  $X_{i+1}^n$  are both decodable at receiver  $i$ . As a result, the achievable rate region of  $(R_i, R_{i+1})$  is bounded by the capacity region of the multiple-access channel  $(X_i^n, X_{i+1}^n, Y_i^n)$ , which is shown in (63). Since (63) reduces to (61) in the strong interference regime, we have shown that (61) is an outer bound of the  $K$ -user cyclic Gaussian interference channel in the strong interference regime. This completes the converse proof.

In the very strong interference regime where  $\text{INR}_i \geq (1 + \text{SNR}_{i-1})\text{SNR}_i, \forall i$ , it is easy to verify that the second constraint in (61) is no longer active. This results in the capacity region (62). ■

#### V. SYMMETRIC CHANNEL AND GENERALIZED DEGREES OF FREEDOM

Consider the symmetric cyclic Gaussian interference channel, where all the direct links from the transmitters to the receivers share the same channel gain and all the cross links share another same channel gain. In addition, all the input signals have the same power constraint  $P$ , i.e.,  $\mathbb{E}[|X_i|^2] \leq P, \forall i$ .

The symmetric capacity of the  $K$ -user interference channel is defined as

$$C_{\text{sym}} = \begin{cases} \text{maximize} & \min\{R_1, R_2, \dots, R_K\} \\ \text{subject to} & (R_1, R_2, \dots, R_K) \in \mathcal{R} \end{cases} \quad (64)$$

where  $\mathcal{R}$  is the capacity region of the  $K$ -user interference channel. For the symmetric interference channel,  $C_{\text{sym}} = \frac{1}{K}C_{\text{sum}}$ , where  $C_{\text{sum}}$  is the sum capacity. As a direct consequence of Theorem 3 and Theorem 5, the generalized degree of freedom of the symmetric capacity for the symmetric cyclic channel can be derived as follows.

<sup>3</sup>This capacity result was also recently obtained in [23].

*Corollary 1:* For the  $K$ -user symmetric cyclic Gaussian interference channel,

$$d_{\text{sym}} = \begin{cases} \min \left\{ \max \left\{ \alpha, 1 - \alpha \right\}, 1 - \frac{\alpha}{2} \right\}, & 0 \leq \alpha < 1 \\ \min \left\{ \frac{\alpha}{2}, 1 \right\}, & \alpha \geq 1 \end{cases} \quad (65)$$

where  $d_{\text{sym}}$  is the generalized degrees of freedom of the symmetric capacity.

Note that the above  $d_{\text{sym}}$  for the  $K$ -user cyclic interference channel with symmetric channel parameters is the same as that of the two-user interference channel derived in [8].

## VI. CONCLUSION

This paper investigates the capacity and the coding strategy for the  $K$ -user cyclic Gaussian interference channel. Specifically, this paper shows that in the weak interference regime, the ETW power-splitting strategy achieves to within two bits of the capacity region. Further, in the special case of  $K = 3$  and with the help of a time-sharing technique, one can achieve to within  $1\frac{1}{2}$  bits of the capacity region in the weak interference regime.

The capacity result for the  $K$ -user cyclic Gaussian interference channel in the strong interference regime is a straightforward extension of the corresponding two-user case. However, in the mixed interference regime, although the constant gap result may well continue to hold, the proof becomes considerably more complicated, as different mixed scenarios need to be enumerated and the corresponding outer bounds derived.

## APPENDIX

### A. Proof of Theorem 1

For the two-user interference channel, Kobayashi and Han [22] gave a detailed Fourier-Motzkin elimination procedure for the achievable rate region. The Fourier-Motzkin elimination for the  $K$ -user cyclic interference channel involves  $K$  elimination steps. The complexity of the process increases with each step. Instead of manually writing down all the inequalities step by step, this appendix uses mathematical induction to derive the final result.

This achievability proof is based on the application of coding scheme in [2] (also referred as the multi-level coding in [24]) to the multi-user setting. Instead of using the original code construction of [1], the following strategy is used in which each common message  $W_i, i = 1, 2, \dots, K$  serves to generate  $2^{nT_i}$  cloud centers  $W_i(j), j = 1, 2, \dots, 2^{nT_i}$ , each of which is surrounded by  $2^{nS_i}$  codewords  $X_i(j, k), k = 1, 2, \dots, 2^{nS_i}$ . This results in achievable rate region expressions expressed in terms of  $(W_i, X_i, Y_i)$  instead of  $(U_i, W_i, Y_i)$ . For the two-user interference channel, Chong, Motani and Garg [2, Lemma 3] made a further simplification to the achievable rate region expression. They observed that in the Han-Kobayashi scheme, the common message  $W_i$  is only required to be correctly decoded at the intended receiver  $Y_i$  and an incorrectly decoded  $W_i$  at receiver  $Y_{i-1}$  does not cause an error event. Based on this observation, they concluded that for the multiple-access channel with input  $(U_i, W_i, W_{i+1})$  and output  $Y_i$ , the rate constraints on common messages  $T_i$ ,

$T_{i+1}$  and  $T_i + T_{i+1}$  are in fact irrelevant to the decoding error probabilities and can be removed, i.e., the rates  $(S_i, T_i, T_{i+1})$  are constrained by only the following set of inequalities:

$$S_i \leq a_i = I(Y_i; X_i | W_i, W_{i+1}, Q) \quad (66)$$

$$S_i + T_i \leq d_i = I(Y_i; X_i | W_{i+1}, Q) \quad (67)$$

$$S_i + T_{i+1} \leq e_i = I(Y_i; W_{i+1}, X_i | W_i, Q) \quad (68)$$

$$S_i + T_i + T_{i+1} \leq g_i = I(Y_i; W_{i+1}, X_i | Q) \quad (69)$$

$$S_i, T_i, T_{i+1} \geq 0 \quad (70)$$

Now, compare the  $K$ -user cyclic interference channel with the two-user interference channel, it is easy to see that in both channel models, each receiver only sees interference from one neighboring transmitter. This makes the decoding error probability analysis for both channel models the same. Therefore, the set of rates  $\mathcal{R}(R_1, R_2, \dots, R_K)$ , where  $R_i = S_i + T_i$ , with  $(S_i, T_i)$  satisfy (66)-(70) for  $i = 1, 2, \dots, K$ , characterizes an achievable rate region for the  $K$ -user cyclic interference channel.

The first step of using the Fourier-Motzkin algorithm is to eliminate all private messages  $S_i$  by substituting  $S_i = R_i - T_i$  into the  $K$  polymatroids (66)-(70). This results in the following  $K$  polymatroids without  $S_i$ :

$$R_i - T_i \leq a_i, \quad (71)$$

$$R_i \leq d_i, \quad (72)$$

$$R_i - T_i + T_{i+1} \leq e_i, \quad (73)$$

$$R_i + T_{i+1} \leq g_i, \quad (74)$$

$$-R_i \leq 0, \quad (75)$$

where  $i = 1, 2, \dots, K$ .

Next, use Fourier-Motzkin algorithm to eliminate common message rates  $T_1, T_2, \dots, T_K$  in a step-by-step process so that after  $n$  steps, common variables  $(T_1, \dots, T_n)$  are eliminated. The induction hypothesis is the following 5 different groups of inequalities, which is assumed to be obtained at the end of the  $n$ th elimination step:

(a) Inequalities not including private or common variables  $S_i$  and  $T_i, i = 1, 2, \dots, K$ :

$$R_i \leq d_i, \quad i = 1, 2, \dots, K \quad (76)$$

$$-R_i \leq 0, \quad i = 1, 2, \dots, n \quad (77)$$

$$R_K + R_1 \leq g_K + a_1, \quad (78)$$

$$R_m \leq a_m + e_{m-1}, \quad (79)$$

$$\sum_{j=l}^m R_j \leq \min \left\{ g_l + \sum_{i=l+1}^{m-1} e_j + a_m, \sum_{j=l-1}^{m-1} e_j + a_m \right\}, \quad (80)$$

$$\sum_{j=1}^m R_j \leq g_1 + \sum_{j=2}^{m-1} e_j + a_m, \quad (81)$$

$$\sum_{j=K}^m R_j \leq g_K + \sum_{j=1}^{m-1} e_j + a_m, \quad (82)$$

where  $m = 2, 3, \dots, n$  and  $l = 2, 3, \dots, m-1$ .

(b) Inequalities including  $T_K$  but not including  $T_{n+1}$ :

$$R_K - T_K \leq a_K, \quad (83)$$

$$-R_K - T_K \leq 0, \quad (84)$$

$$-T_K \leq 0, \quad (85)$$

$$\sum_{j=K}^p R_j - T_K \leq \sum_{j=K}^{p-1} e_j + a_p, \quad (86)$$

where  $p = 1, 2, \dots, n$ .

(c) All other inequalities not including  $T_{n+1}$ :

$$R_{n+1} + T_{n+2} \leq g_{n+1}, \quad (87)$$

and all the polymatroids in (71)-(75) indexed from  $n+2$  to  $K-1$ .

(d) Inequalities including  $T_{n+1}$  with a plus sign:

$$T_{n+1} \leq e_n, \quad (88)$$

$$-R_{n+1} + T_{n+1} \leq 0, \quad (89)$$

$$\sum_{j=l}^n R_j + T_{n+1} \leq \min \left\{ \sum_{j=l-1}^n e_j, g_l + \sum_{j=l+1}^n e_j \right\}, \quad (90)$$

$$\sum_{j=1}^n R_j + T_{n+1} \leq g_1 + \sum_{j=2}^n e_j, \quad (91)$$

$$\sum_{j=K}^n R_j + T_{n+1} \leq g_K + \sum_{j=1}^n e_j, \quad (92)$$

$$\sum_{j=K}^n R_j + T_{n+1} - T_K \leq \sum_{j=K}^n e_j, \quad (93)$$

where  $l$  goes from 2 to  $n$ .

(e) Inequalities including  $T_{n+1}$  with a minus sign:

$$R_{n+1} - T_{n+1} \leq a_{n+1}, \quad (94)$$

$$R_{n+1} - T_{n+1} + T_{n+2} \leq e_{n+1}, \quad (95)$$

$$-T_{n+1} \leq 0. \quad (96)$$

It is easy to verify the correctness of inequalities (76)-(96) for  $n = 2$ . We next show that for  $n < K - 2$ , if at the end of step  $n$ , the inequalities in (76)-(96) hold, then they must also hold at the end of step  $n+1$ . Towards this end, we follow the Fourier-Motzkin algorithm [22] by first adding up all the inequalities in (88)-(93) with each of the inequalities in (94)-(96) to eliminate  $T_{n+1}$ . This results in the following three groups of inequalities:

(a) Inequalities due to (94):

$$R_{n+1} \leq a_{n+1} + e_n, \quad (97)$$

$$0 \leq a_{n+1}, \quad (98)$$

$$\sum_{j=l}^{n+1} R_j \leq \min \left\{ \sum_{j=l-1}^n e_j + a_{n+1}, g_l + \sum_{j=l+1}^n e_j + a_{n+1} \right\}, \quad (99)$$

$$\sum_{j=1}^{n+1} R_j \leq g_1 + \sum_{j=2}^n e_j + a_{n+1}, \quad (100)$$

$$\sum_{j=K}^{n+1} R_j \leq g_K + \sum_{j=1}^n e_j + a_{n+1}, \quad (101)$$

$$\sum_{j=K}^{n+1} R_j - T_K \leq \sum_{j=K}^n e_j + a_{n+1}, \quad (102)$$

where  $l = 2, 3, \dots, n$ .

(b) Inequalities due to (95):

$$R_{n+1} + T_{n+2} \leq e_n + e_{n+1}, \quad (103)$$

$$T_{n+2} \leq e_{n+1}, \quad (104)$$

$$\sum_{j=l}^{n+1} R_j + T_{n+2} \leq \min \left\{ \sum_{j=l-1}^{n+1} e_j, g_l + \sum_{j=l+1}^{n+1} e_j \right\}, \quad (105)$$

$$\sum_{j=1}^{n+1} R_j + T_{n+2} \leq g_1 + \sum_{j=2}^{n+1} e_j, \quad (106)$$

$$\sum_{j=K}^{n+1} R_j + T_{n+2} \leq g_K + \sum_{j=1}^{n+1} e_j, \quad (107)$$

$$\sum_{j=K}^{n+1} R_j + T_{n+2} - T_K \leq \sum_{j=K}^{n+1} e_j, \quad (108)$$

where  $l = 2, 3, \dots, n$ .

(c) Inequalities due to (96):

$$0 \leq e_n, \quad (109)$$

$$-R_{n+1} \leq 0, \quad (110)$$

$$\sum_{j=l}^n R_j \leq \min \left\{ \sum_{j=l-1}^n e_j, g_l + \sum_{j=l+1}^n e_j \right\} \quad (111)$$

$$\sum_{j=1}^n R_j \leq g_1 + \sum_{j=2}^n e_j, \quad (112)$$

$$\sum_{j=K}^n R_j \leq g_K + \sum_{j=1}^n e_j, \quad (113)$$

$$\sum_{j=K}^n R_j - T_K \leq \sum_{j=K}^n e_j, \quad (114)$$

where  $l = 2, 3, \dots, n$ .

Inspecting the above three groups of inequalities, we can see that (98) and (109) are obviously redundant. Also, (111)



is redundant due to (80), (112) is redundant due to (81), (113) is redundant due to (82), and (114) is redundant due to (86). Now, with these six redundant inequalities removed, the above three groups of inequalities in (97)-(110) together with (76)-(87) form the set of inequalities at the end of step  $n+1$ . It can be verified that this new set of inequalities is exactly (76)-(96) with  $n$  replaced by  $n+1$ . This completes the induction part.

Now, we proceed with the  $(K-1)$ th step. At the end of this step,  $T_1, T_2, \dots, T_{K-1}$  would all be removed and only  $T_K$  would remain. Because of the cyclic nature of the channel, the set of inequalities (76)-(96) needs to be modified for this  $n = K-1$  case. It can be verified that at the end of the  $(K-1)$ th step of Fourier-Motzkin algorithm, we obtain the following set of inequalities:

(a) Inequalities not including  $T_K$ : (76)-(82) with  $n$  replaced by  $K-1$  and

$$\sum_{j=1}^K R_j \leq \sum_{j=1}^K e_j. \quad (115)$$

(b) Inequalities including  $T_K$  with a plus sign: (88)-(92) with  $n$  replace by  $K-1$ . Note that, (93) becomes (115) when  $n = K-1$ .

(c) Inequalities including  $T_K$  with a minus sign:

$$R_K - T_K \leq a_K, \quad (116)$$

$$\sum_{j=K}^l R_j - T_K \leq \sum_{j=K}^{l-1} e_j + a_l, \quad (117)$$

$$-T_K \leq 0, \quad (118)$$

where  $l = 1, 2, \dots, K-1$ .

In the  $K$ th step (final step) of the Fourier-Motzkin algorithm,  $T_K$  is eliminated by adding each of the inequalities involving  $T_K$  with a plus sign and each of the inequalities involving  $T_K$  with a minus sign to obtain new inequalities not involving  $T_K$ . (This is quite similar to the procedure of obtaining (97)-(114).) Finally, after removing all the redundant inequalities, we obtain the set of inequalities in Theorem 1.

## B. Proof of Theorem 2

We will prove the outer bounds from (20) to (23) one by one.

First, (20) is simply the cut-set upper bound for user  $i$ .

Second, (21) is the bound on the sum-rate of  $l$  adjacent users starting from  $m$ . According to Fano's inequality, for a

block of length  $n$ , we have

$$\begin{aligned} & n \left( \sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \\ & \leq \sum_{j=m}^{m+l-1} I(x_j^n; y_j^n) \\ & \stackrel{(a)}{\leq} h(y_m^n) - h(y_m^n | x_m^n) + \sum_{j=m+1}^{m+l-2} I(x_j^n; y_j^n s_j^n) \\ & \quad + I(x_{m+l-1}^n; y_{m+l-1}^n | x_{m+l}^n) \\ & = h(y_m^n) - h(s_{m+1}^n) \\ & \quad + \sum_{j=m+1}^{m+l-2} [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(z_{m+l-1}^n) \\ & = h(y_m^n) - h(z_{m+l-1}^n) + \sum_{j=m+1}^{m+l-2} [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) \\ & \quad - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n) \\ & \stackrel{(b)}{\leq} n \left( \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1} \right), \quad (119) \end{aligned}$$

where in (a) we give genie  $s_j^n$  to  $y_j^n$  for  $m+1 \leq j \leq m+l-2$  and  $x_{m+l}^n$  to  $y_{m+l-1}^n$  (genies  $s_j^n$  are as defined in [25, Theorem 2]), and (b) comes from the fact [8] that Gaussian inputs maximize 1) entropy  $h(y_m^n)$ , 2) conditional entropy  $h(y_j^n | s_j^n)$  for any  $j$ , and 3) entropy difference  $h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n)$ . This proves the first bound in (21).

Similarly, the second upper bound of (21) can be obtained by giving genie  $s_j^n$  to  $y_j^n$  for  $m \leq j \leq m+l-2$  and  $x_{m+l}^n$  to  $y_{m+l-1}^n$ :

$$\begin{aligned} & n \left( \sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \\ & \leq \sum_{j=m}^{m+l-1} I(x_j^n; y_j^n) \\ & \leq \sum_{j=m}^{m+l-2} I(x_j^n; y_j^n s_j^n) + I(x_{m+l-1}^n; y_{m+l-1}^n | x_{m+l}^n) \\ & = \sum_{j=m}^{m+l-2} [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(z_{m+l-1}^n) \\ & = h(s_m^n) - h(z_{m+l-1}^n) + \sum_{j=m}^{m+l-2} [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) \\ & \quad - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n) \\ & \leq n \left( \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right). \quad (120) \end{aligned}$$

Combining (119) and (120) gives the upper bound in (21).

Third, the first upper bound in (22) is in fact the non-symmetric version of [25, Theorem 2], from which we have

$$\begin{aligned} R_{\text{sum}} - n\epsilon_n &\leq \sum_{k=1}^K \{h(y_{ki}|s_{ki}) - h(z_{ki})\} \\ &\leq n \sum_{j=1}^K \alpha_j. \end{aligned} \quad (121)$$

The other sum-rate upper bounds (i.e.,  $\rho_l$ ) can be derived by giving genies  $x_l^n$  to  $y_{l-1}^n$  and  $s_j^n$  to  $y_j^n$  for  $j = 1, 2, \dots, K, j \neq l, l-1$ :

$$\begin{aligned} n(R_{\text{sum}} - \epsilon_n) &\leq I(x_1^n; y_1^n) + I(x_2^n; y_2^n) + \dots + I(x_K^n; y_K^n) \\ &= I(x_{l-1}^n; y_{l-1}^n | x_l^n) + I(x_l^n; y_l^n) + \sum_{j=1, j \neq l, l-1}^K I(x_j^n; y_j^n | s_j^n) \\ &= h(h_{l-1, l-1} x_{l-1}^n + z_{l-1}^n) - h(z_{l-1}^n) + h(y_l^n) - h(s_{l+1}^n) \\ &\quad + \sum_{j=1, j \neq l, l-1}^K [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ &= h(y_l^n) - h(z_{l-1}^n) + h(h_{l-1, l-1} x_{l-1}^n + z_{l-1}^n) \\ &\quad - h(h_{l-1, l-2} x_{l-1}^n + z_{l-2}^n) \\ &\quad + \sum_{j=1, j \neq l, l-1}^K [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ &\leq n \left( \beta_{l-1} + \gamma_l + \sum_{j=1, j \neq l, l-1}^K \alpha_j \right) \\ &= n\rho_l \end{aligned} \quad (122)$$

where  $l = 1, 2, \dots, K$ .

Fourth, for the bound in (23), from Fano's inequality, we have

$$\begin{aligned} n(R_{\text{sum}} + R_i - \epsilon_n) &\leq \sum_{j=1}^K I(x_j^n; y_j^n) + I(x_i^n; y_i^n) \\ &\stackrel{(a)}{\leq} I(x_i^n; y_i^n) + I(x_i^n; y_i^n | x_{i+1}^n) + \sum_{j=1, j \neq i}^K I(x_j^n; y_j^n | s_j^n) \\ &= h(y_i^n) - h(s_{i+1}^n) + h(h_{i, i} x_i^n + z_i^n) - h(z_i^n) \\ &\quad + \sum_{j=1, j \neq i}^K [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ &= h(y_i^n) - h(z_i^n) + h(h_{i, i} x_i^n + z_i^n) - h(h_{i, i-1} x_i^n + z_i^n) \\ &\quad + \sum_{j=1, j \neq i}^K [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ &\leq n \left( \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j \right) \end{aligned} \quad (123)$$

where in (a) we give genie  $x_{i+1}^n$  to  $y_i^n$  and  $s_j^n$  to  $y_j^n$  for  $j = 1, 2, \dots, K, j \neq i$ .

C. Proof of  $\mathcal{R}_{\text{HK-TS}}^{(3)} \subseteq \mathcal{R}_{\text{HK}}^{(3)}$

For a fixed  $P_3 \subseteq \mathcal{P}_3$ , define

$$P_3^* = \sum_{w_1} P_3, \quad P_3^{**} = \sum_{w_2} P_3, \quad P_3^{***} = \sum_{w_3} P_3. \quad (124)$$

We will show that

$$\begin{aligned} \mathcal{R}_{\text{HK-TS}}^{(3)}(P_3) &\subseteq \mathcal{R}_{\text{HK}}^{(3)}(P_3) \cup \mathcal{R}_{\text{HK}}^{(3)}(P_3^*) \cup \mathcal{R}_{\text{HK}}^{(3)}(P_3^{**}) \cup \mathcal{R}_{\text{HK}}^{(3)}(P_3^{***}). \end{aligned} \quad (125)$$

Suppose that rate triple  $(R_1, R_2, R_3)$  is in  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$  but not in  $\mathcal{R}_{\text{HK}}^{(3)}(P_3)$ . Then at least one of the following inequalities hold:

$$a_1 + e_3 \leq R_1 \leq d_1, \quad (126)$$

$$a_2 + e_1 \leq R_2 \leq d_2, \quad (127)$$

$$a_3 + e_2 \leq R_3 \leq d_3, \quad (128)$$

Without loss of generality, assume that (126) holds.

Substituting  $W_1 = \emptyset$  into  $\mathcal{R}_{\text{HK}}^{(3)}(P_3)$ , we obtain  $\mathcal{R}_{\text{HK}}^{(3)}(P_3^*)$  as follows:

$$R_1 \leq d_1, \quad (129)$$

$$R_2 \leq \min\{d_2, a_2 + g_1\}, \quad (130)$$

$$R_3 \leq \min\{I(Y_3; X_3 | Q), e_2 + I(Y_3; X_3 | W_3, Q)\}, \quad (131)$$

$$R_1 + R_2 \leq a_2 + g_1, \quad (132)$$

$$R_2 + R_3 \leq \min\{g_2 + I(Y_3; X_3 | W_3, Q), g_1 + e_2 + I(Y_3; X_3 | W_3, Q)\}, \quad (133)$$

$$R_3 + R_1 \leq \min\{d_1 + I(Y_3; X_3 | Q), d_1 + e_2 + I(Y_3; X_3 | W_3, Q)\}, \quad (134)$$

$$R_1 + R_2 + R_3 \leq g_1 + e_2 + I(Y_3; X_3 | W_3, Q). \quad (135)$$

We will show that whenever (126) is true, we have  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3) \subseteq \mathcal{R}_{\text{HK}}^{(3)}(P_3^*)$ . To this end, inspect  $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$  in (47)-(54). From (47), we have

$$R_1 \leq d_1, \quad (136)$$

and from (47) and (126) and (48), we have

$$\begin{aligned} R_2 &\leq \min\{d_2, a_2 + e_1 - a_1\} \\ &\leq \min\{d_2, a_2 + g_1\}, \end{aligned} \quad (137)$$

and from (126) and (50), we have

$$\begin{aligned} R_3 &\leq \min\{g_3 - e_3, e_2\} \\ &\leq \min\{I(Y_3; X_3 | Q), e_2 + I(Y_3; X_3 | W_3, Q)\}, \end{aligned} \quad (138)$$

and from (48), we have

$$R_1 + R_2 \leq a_2 + g_1, \quad (139)$$

and from (126) and (51), we have

$$\begin{aligned} R_2 + R_3 &\leq \min\{g_2, e_1 + e_2 - a_1\} \\ &\leq \min\{g_2 + I(Y_3; X_3 | W_3, Q), g_1 + e_2 + I(Y_3; X_3 | W_3, Q)\}, \end{aligned} \quad (140)$$

and from (126) and (50), we have

$$\begin{aligned} R_3 + R_1 &\leq \min\{d_1 + g_3 - a_3, e_2 + d_1\} \\ &\leq \min\{d_1 + I(Y_3; X_3|Q), \\ &\quad d_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (141)$$

and from (126) and (52), we have

$$\begin{aligned} R_1 + R_2 + R_3 &\leq g_1 + e_2 \\ &\leq g_1 + e_2 + I(Y_3; X_3|W_3, Q). \end{aligned} \quad (142)$$

It is easy to see that  $(R_1, R_2, R_3)$  satisfying the above constrains (136)-(142) is within the rate region  $\mathcal{R}_{\text{HK}}^{(3)}(P_3^*)$ . In the same way, we can prove the cases for when (127) holds and when (128) holds.

Therefore, (125) is true, and it immediately follows that

$$\mathcal{R}_{\text{HK-TS}}^{(3)} \subseteq \mathcal{R}_{\text{HK}}^{(3)}. \quad (143)$$

#### D. Useful Inequalities

Keep in mind that, with the ETW's power splitting strategy, i.e.,  $\text{SNR}_{ip} = \min\{\text{SNR}_i, \frac{\text{SNR}_i}{1 + \text{INR}_i}\}$ , we always have  $\text{SNR}_{ip} > \frac{\text{SNR}_i}{1 + \text{INR}_i}$ . This appendix presents several useful inequalities as follows. For all  $i = 1, 2, \dots, K$ ,

- $\lambda_i - d_i < 1$ , because

$$\begin{aligned} \lambda_i - d_i &= \log(1 + \text{SNR}_i) - \log(2 + \text{SNR}_i) + 1 \\ &= 1 - \log\left(\frac{2 + \text{SNR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 1 \end{aligned} \quad (144)$$

- $\lambda_i - (a_i + e_{i-1}) < 2$ , because

$$\begin{aligned} \lambda_i - (a_i + e_{i-1}) &= \log(1 + \text{SNR}_i) - \log(2 + \text{SNR}_{ip}) + 1 \\ &\quad - \log(1 + \text{INR}_i + \text{SNR}_{i-1,p}) + 1 \\ &< 2 + \log(1 + \text{SNR}_i) - \log\left(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) \\ &\quad - \log(1 + \text{INR}_i) \\ &= 2 - \log\left(1 + \frac{\text{INR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 2 \end{aligned} \quad (145)$$

- $\beta_i - a_i < 1$ , because

$$\begin{aligned} \beta_i - a_i &= \log\left(\frac{1 + \text{SNR}_i}{1 + \text{INR}_i}\right) - \log(2 + \text{SNR}_{ip}) + 1 \\ &< \log\left(\frac{1 + \text{SNR}_i}{1 + \text{INR}_i}\right) - \log\left(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) + 1 \\ &= 1 - \log\left(1 + \frac{\text{INR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 1 \end{aligned} \quad (146)$$

- $\alpha_i - e_i < 1$ , because

$$\begin{aligned} \alpha_i - e_i &= \log\left(1 + \text{INR}_{i+1} + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) \\ &\quad - \log(1 + \text{INR}_{i+1} + \text{SNR}_{ip}) + 1 \\ &\leq 1 \end{aligned} \quad (147)$$

- $\gamma_i - g_i = 1$ , because

$$\begin{aligned} \gamma_i - g_i &= \log(1 + \text{INR}_{i+1} + \text{SNR}_i) \\ &\quad - \log(1 + \text{INR}_{i+1} + \text{SNR}_i) + 1 \\ &= 1 \end{aligned} \quad (148)$$

- $\mu_i - e_{i-1} < 1$ , because

$$\begin{aligned} \mu_i - e_{i-1} &= \log(1 + \text{INR}_i) \\ &\quad - \log(1 + \text{INR}_i + \text{SNR}_{i-1,p}) + 1 \\ &\leq 1 \end{aligned} \quad (149)$$

#### REFERENCES

- [1] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Trans. Inf. Theory*, vol. 27, no. 1, pp. 49–60, Jan. 1981.
- [2] H. Chong, M. Motani, H. Garg, and H. El Gamal, "On the Han-Kobayashi region for the interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 7, pp. 3188–3195, Jul. 2008.
- [3] A. B. Carleial, "Interference channels," *IEEE Trans. Inf. Theory*, vol. 24, no. 1, pp. 60–70, Jan. 1978.
- [4] H. Sato, "The capacity of the Gaussian interference channel under strong interference," *IEEE Trans. Inf. Theory*, vol. 27, no. 6, pp. 786–788, Nov. 1981.
- [5] V. S. Annapureddy and V. Veeravalli, "Gaussian interference networks: Sum capacity in the low interference regime and new outer bounds on the capacity region," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3032–3035, Jul. 2009.
- [6] A. S. Motahari and A. K. Khandani, "Capacity bounds for the Gaussian interference channel," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 620–643, Feb. 2009.
- [7] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689–699, Feb. 2009.
- [8] R. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.
- [9] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the K user interference channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.
- [10] G. Bresler, A. Parekh, and D. N. C. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," *IEEE Trans. Inf. Theory*, vol. 56, no. 9, pp. 4566–4592, Sep. 2010.
- [11] X. Shang, G. Kramer, and B. Chen, "New outer bounds on the capacity region of Gaussian interference channels," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Jul. 2008, pp. 245–249.
- [12] D. Tuninetti, "A new sum-rate outer bound for interference channels with three source-destination pairs," in *Proc. Inf. Theory and App. (ITA)*, Feb. 2011, pp. 1–8.
- [13] A. Chaaban and A. Sezgin, "The capacity region of the 3-user Gaussian interference channel with mixed strong-very strong interference," in *Proc. Int. ITG Workshop on Smart Antennas (WSA)*, Feb. 2011, pp. 1–5.
- [14] S. A. Jafar and S. Vishwanath, "Generalized degrees of freedom of the symmetric Gaussian K user interference channel," *IEEE Trans. Inf. Theory*, vol. 56, no. 7, pp. 3297–3303, July. 2010.
- [15] V. R. Cadambe and S. A. Jafar, "Interference alignment and a noisy interference regime for many-to-one interference channels," *Submitted to IEEE Trans. Inf. Theory*, Dec. 2009. [Online]. Available: <http://arxiv.org/pdf/0912.3029>
- [16] O. Ordentlich, U. Erez, and B. Nazer, "The approximate sum capacity of the symmetric Gaussian k-user interference channel," *Submitted to IEEE Trans. Inf. Theory*, Jun. 2012. [Online]. Available: <http://arxiv.org/abs/1206.0197>
- [17] O. Somekh, B. M. Zaidel, and S. Shamai, "Sum rate characterization of joint multiple cell-site processing," *IEEE Trans. Inf. Theory*, vol. 53, no. 12, pp. 4473–4497, Dec. 2007.
- [18] A. D. Wyner, "Shannon-theoretic approach to a Gaussian cellular multiple-access channel," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 1713–1727, Nov. 1994.
- [19] Y. Liang and A. Goldsmith, "Symmetric rate capacity of cellular systems with cooperative base stations," in *Proc. Global Telecommun. Conf. (GLOBECOM)*, Nov. 2006, pp. 1–5.

- [20] J. Sheng, D. N. C. Tse, J. Hou, J. B. Soriaga, and R. Padovani, "Multi-cell downlink capacity with coordinated processing," in *Proc. Inf. Theory and App. (ITA)*, Jan. 2007, pp. 1–5.
- [21] E. Sasoglu, "Successive cancellation for cyclic interference channels," in *Proc. IEEE Inf. Theory Workshop (ITW)*, May 2008, pp. 36–40.
- [22] K. Kobayashi and T. S. Han, "A further consideration on the HK and the CMG regions for the interference channel," in *Proc. Inf. Theory and App. (ITA)*, Jan. 2007.
- [23] A. Chaaban and A. Sezgin, "On the capacity of a class of multi-user interference channels," in *Proc. Int. ITG Workshop on Smart Antennas (WSA)*, Feb. 2011, pp. 1–5.
- [24] A. Raja, V. M. Prabhakaran, and P. Viswanath, "The two-user compound interference channel," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5100–5120, Nov. 2009.
- [25] L. Zhou and W. Yu, "On the symmetric capacity of the k-user symmetric cyclic Gaussian interference channel," in *Proc. IEEE Conf. Inf. Sciences and Systems (CISS)*, Mar. 2010, pp. 1–6.

**Lei Zhou** (S'05) received the B.E. degree in electronics engineering from Tsinghua University, Beijing, China, in 2003 and M.A.Sc. degree in electrical and computer engineering from the University of Toronto, ON, Canada, in 2008. During 2008-2009, he was with Nortel Networks, Ottawa, ON, Canada. He is currently pursuing the Ph.D. degree with the Department of Electrical and Computer Engineering, University of Toronto, Canada. His research interests include multiterminal information theory, wireless communications, and signal processing.

He is a recipient of the Shahid U.H. Qureshi Memorial Scholarship in 2011, the Alexander Graham Bell Canada Graduate Scholarship for 2011-2013, and the Chinese government award for outstanding self-financed students abroad in 2012.

**Wei Yu** (S'97-M'02-SM'08) received the B.A.Sc. degree in Computer Engineering and Mathematics from the University of Waterloo, Waterloo, Ontario, Canada in 1997 and M.S. and Ph.D. degrees in Electrical Engineering from Stanford University, Stanford, CA, in 1998 and 2002, respectively. Since 2002, he has been with the Electrical and Computer Engineering Department at the University of Toronto, Toronto, Ontario, Canada, where he is now Professor and holds a Canada Research Chair in Information Theory and Digital Communications. His main research interests include multiuser information theory, optimization, wireless communications and broadband access networks.

Prof. Wei Yu currently serves as an Associate Editor for IEEE TRANSACTIONS ON INFORMATION THEORY. He was an Editor for IEEE TRANSACTIONS ON COMMUNICATIONS (2009-2011), an Editor for IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS (2004-2007), and a Guest Editor for a number of special issues for the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS and the EURASIP JOURNAL ON APPLIED SIGNAL PROCESSING. He is member of the Signal Processing for Communications and Networking Technical Committee of the IEEE Signal Processing Society. He received the IEEE Signal Processing Society Best Paper Award in 2008, the McCharles Prize for Early Career Research Distinction in 2008, the Early Career Teaching Award from the Faculty of Applied Science and Engineering, University of Toronto in 2007, and the Early Researcher Award from Ontario in 2006.